MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

42nd Annual

AIME I

American Invitational Mathematics Examination I

Thursday, February 1, 2024

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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Problem 1:

Every morning Aya goes for a 9-kilometer-long walk and stops at a coffee shop afterwards. When she walks at a constant speed of s kilometers per hour, the walk takes her 4 hours, including t minutes spent in the coffee shop. When she walks at $s+2$ kilometers per hour, the walk takes her 2 hours and 24 minutes, including t minutes spent in the coffee shop. Suppose Aya walks at $s+\frac{1}{2}$ kilometers per hour. Find the number of minutes the walk takes her, including the t minutes spent in the coffee shop.

Solution:

Answer (204):

Let T be the time in hours that Aya spends in the coffee shop. Then $\frac{9}{s}+T=4$, and $\frac{9}{s+2}+T=2\frac{2}{5}$. Eliminating T gives $0=4s^2+8s-45=(2s-5)(2s+9)$. Thus $s=\frac{5}{2}$ and

$$T=4-\frac{9}{\frac{5}{2}}=\frac{2}{5}.$$

The requested number of minutes the walk takes Aya is therefore

$$60\cdot\left(\frac{9}{\frac{5}{2}+\frac{1}{2}}+\frac{2}{5}\right)=204.$$

Problem 2:

There exist real numbers x and y, both greater than 1, such that $\log_x\left(y^x\right)=\log_y\left(x^{4y}\right)=10$. Find xy.

Solution:

Answer (025):

The given equations imply that

$$x\log_x y=4y\log_y x=10.$$

Multiplying the two expressions equaling 10 yields

$$100=10\cdot 10=(x\log_x y)\cdot\left(4y\log_y x\right)=4xy\cdot\log_x y\cdot\log_y x=4xy.$$

Thus $xy=25$.

OR

Converting the given equations from logarithmic form into exponential form gives

$$x^{10}=y^x$$
$$y^{10}=x^{4y}.$$

Raising each side of the second equation to the power x and using the first equation yields

$$x^{4xy} = y^{10x} = \left(y^x\right)^{10} = \left(x^{10}\right)^{10} = x^{100}.$$

Thus $4xy = 100$, so $xy = 25$.

Note: The equations are satisfied by the real numbers $x \approx 1.535$ and $y \approx 16.291$.

Problem 3:

Alice and Bob play the following game. A stack of n tokens lies before them. The players take turns with Alice going first. On each turn, the player removes either 1 token or 4 tokens from the stack. Whoever removes the last token wins. Find the number of positive integers n less than or equal to 2024 for which there exists a strategy for Bob that guarantees that Bob will win the game regardless of Alice's play.

Solution:

Answer (809):

Assume that both Alice and Bob play optimally. Then Alice will win if $n = 1$. If $n = 2$, then Alice must remove 1 token, leaving a winning position for Bob. If $n = 3$, then Alice must remove 1 token, leaving a losing position for Bob, so Alice will win. If $n = 4$, then Alice can take all the tokens and win. If $n = 5$, then whether Alice takes 1 token or 4 tokens, she leaves a winning position for Bob. Thus for $1 \le n \le 5$, if $n \equiv 1, 3,$ or $4 \pmod 5$, then the player presented with that position can win, but if $n \equiv 0$ or $2 \pmod 5$, then the player presented with that position will lose if the other player plays optimally.

By induction, Alice has a winning strategy exactly when positive integer n is congruent to 1, 3, or 4 (mod 5). Indeed, suppose this is true for all positive integers less than some $k > 5$. If $k \equiv 1$ or $3 \pmod 5$, then a player presented with that position can remove 1 token and present the opponent with a losing position, with the number of tokens congruent to 0 or 2 (mod 5), respectively. If $k \equiv 4 \pmod 5$, then a player presented with that position can remove 4 tokens and present the opponent with a losing position, with the number of tokens congruent to 0 (mod 5). On the other hand, if $k \equiv 0$ or $2 \pmod 5$, then removing 1 or 4 tokens will present the opponent with a number of tokens congruent to 1, 3, or 4 (mod 5), from which the opponent can win. Thus Alice can guarantee winning the game if $k \equiv 1, 3,$ or $4 \pmod 5$, and Bob can guarantee winning if $k \equiv 0$ or $2 \pmod 5$.

The number of positive integers less than or equal to 2024 congruent to 0 or 2 (mod 5) is $\frac{2}{5} \cdot 2020 + 1 = 809$.

Problem 4:

Jen enters a lottery by selecting four distinct elements of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then four distinct elements of S are drawn at random. Jen wins a prize if at least two of her numbers are drawn, and she wins the grand prize if all four of her numbers are drawn. The probability that Jen wins the grand prize given that Jen wins a prize is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

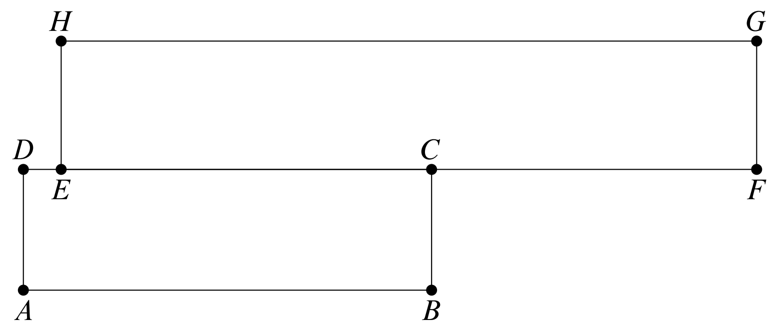
Solution:

Answer (116):

There is only 1 way for Jen to win the grand prize, and that is for the drawn numbers to match all four of her numbers. The number of ways to match exactly 2 of the 4 numbers drawn is $\binom{4}{2} \cdot \binom{10-4}{2} = 90$. The number of ways to match exactly 3 of the 4 numbers drawn is $\binom{4}{3} \cdot \binom{10-4}{1} = 24$. Thus there are $1 + 90 + 24 = 115$ equally likely ways to win a prize. The probability that Jen wins the grand prize given that Jen wins a prize is therefore $\frac{1}{115}$. The requested sum is $1 + 115 = 116$.

Problem 5:

Rectangle $ABCD$ has dimensions $AB = 107$ and $BC = 16$, and rectangle $EFGH$ has dimensions $EF = 184$ and $FG = 17$. Points D, E, C, and F lie on line DF in that order, and A and H lie on opposite sides of line DF, as shown. Points A, D, H, and G lie on a common circle. Find CE.



Solution:

Answer (104):

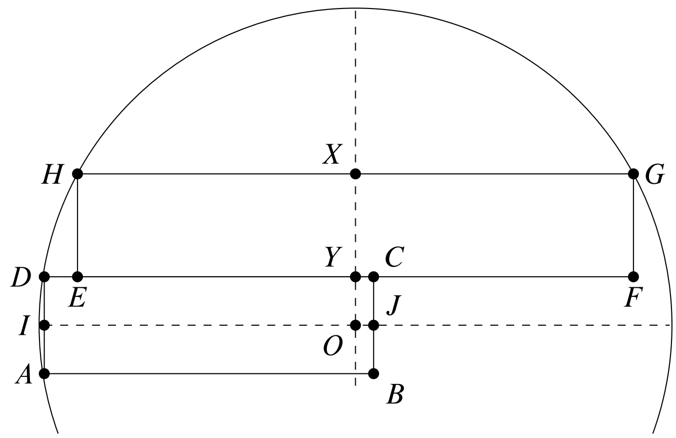
Let W be the intersection of lines GH and AD. Let $z = DE$. Then the power of the point W with respect to the circle passing through A, D, H, and G is $WD \cdot WA = WH \cdot WG$, so $17 \cdot (17 + 16) = z(z + 184)$, which simplifies to

$$0 = z^2 + 184z - 561 = (z + 187)(z - 3).$$

Thus $z = 3$, and the requested distance is $CE = 107 - z = 104$.

OR

Let I, J, X, and Y be the midpoints of $\overline{AD}$, $\overline{BC}$, $\overline{GH}$, and $\overline{EF}$, respectively, and let O be the center of the circle circumscribing $ADHG$. Then lines IJ and XY intersect at point O, as shown.



Then $DI = 8$, $OX = 25$, and $HX = 92$. Let $x = CE$. Then $DE = CD - x = 107 - x$, and

$$OI = HX + DE = 199 - x.$$

Applying the Pythagorean Theorem to right triangles $\triangle OID$ and $\triangle OXH$ gives

$$OD^2 = (199 - x)^2 + 8^2 \quad \text{and} \quad OH^2 = 25^2 + 92^2.$$

The lengths of OD and OH are equal because they are radii of the same circle. Hence

$$(199 - x)^2 = 92^2 - 8^2 + 25^2 = (92 - 8)(92 + 8) + 25^2 = 336 \cdot 25 + 25^2 = 25 \cdot 361 = 5^2 \cdot 19^2 = 95^2.$$

The length x must be less than CD, which is given to be 107, so $199 - x$ is a positive real number. Thus $199 - x = 95$, and $x = 199 - 95 = 104$.

OR

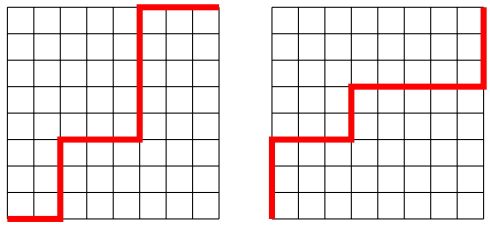
Let $x = CE$. Because quadrilateral $ADHG$ is cyclic, $\angle ADH$ and $\angle AGH$ are supplementary. Let W be the intersection of lines AD and GH. Then $\triangle AGW \sim \triangle DHE$, so $\frac{DE}{EH} = \frac{AW}{WG}$, which implies

$$\frac{107 - x}{17} = \frac{33}{184 + 107 - x} = \frac{33}{291 - x}.$$

Thus $(107 - x)(291 - x) = 17 \cdot 33$. Substituting $y = 199 - x$ yields $(y - 92)(y + 92) = 561$, so $y = 95$. The requested distance is $CE = x = 199 - y = 199 - 95 = 104$.

Problem 6:

Consider the paths of length 16 that follow the lines from the lower left corner to the upper right corner on an 8×8 grid. Find the number of such paths that change direction exactly four times, as in the examples shown below.



Solution:

Answer (294):

Any such path must consist of 8 steps to the right and 8 steps upward in some order. First consider those paths whose first step is to the right. All such paths start and end with a horizontal segment and include two vertical segments with a horizontal segment in between. The path is completely determined by the choice of the positions of the two vertical segments and the height of the middle horizontal segment. There are $\binom{7}{2} = 21$ ways to select the positions of the two vertical segments and 7 ways to select the height of the middle horizontal segment. Thus there are $21 \cdot 7 = 147$ paths that begin with a horizontal step. By symmetry, there are also 147 paths that begin with a vertical step, so the total number of paths is $2 \cdot 147 = 294$.

Problem 7:

Find the greatest possible real part of

$$(75 + 117i)z + \frac{96 + 144i}{z},$$

where z is a complex number with $|z| = 4$. Here $i = \sqrt{-1}$.

Solution:

Answer (540):

Note that

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{16}.$$

There is a real number θ such that $z = 4\cos\theta + 4i\sin\theta$, so

$$(75 + 117i)z + \frac{96 + 144i}{z} = (75 + 117i)z + (6 + 9i)\overline{z}$$

has real part $75 \cdot 4\cos\theta - 117 \cdot 4\sin\theta + 6 \cdot 4\cos\theta + 9 \cdot 4\sin\theta = 4(81\cos\theta - 108\sin\theta)$.

Let ϕ be a real number with $\sin\phi = \frac{108}{135} = \frac{4}{5}$. Then

$$81\cos\theta - 108\sin\theta = 135\left(\frac{81}{135}\cos\theta - \frac{108}{135}\sin\theta\right) = 135\cos(\theta + \phi).$$

Thus the expression has a maximum of 135 when $\theta = -\phi$.

Alternatively, by the Cauchy-Schwarz Inequality,

$$81\cos\theta - 108\sin\theta \le \sqrt{\left(\cos^2\theta + \sin^2\theta\right)\left(81^2 + 108^2\right)} = 135,$$

with equality occurring when $\sin\theta = -\frac{108}{135} = -\frac{4}{5}$.
Hence the requested greatest real part is $4 \cdot 135 = 540$.

OR

As in the first solution,

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{\overline{z}}{16},$$

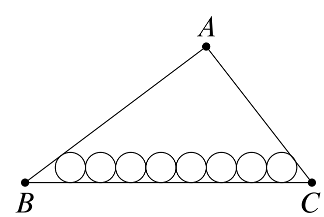
so the given expression can be written as $(75 + 117i)z + (6 + 9i)\overline{z}$. The real part of a complex number is not affected by conjugation, so the real part the given expression is the same as the real part of

$$(75 + 117i)z + \overline{(6 + 9i)\overline{z}} = (75 + 117i)z + (6 - 9i)z = (81 + 108i)z.$$

As z varies on the circle of radius 4 centered at the origin, the number $(81 + 108i)z$ rotates around a circle centered at the origin with radius $|z| \cdot \sqrt{81^2 + 108^2} = 4 \cdot 9\sqrt{9^2 + 12^2} = 540$. Thus the given expression has a maximum real part for the value of z corresponding to the point of this circle on the positive real axis, and that real part is 540.

Problem 8:

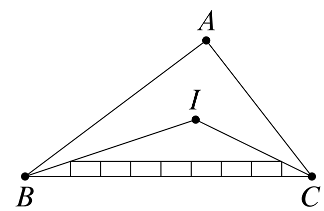
Eight circles of radius 34 can be placed tangent to side $\overline{BC}$ of $\triangle ABC$ so that the circles are sequentially tangent to each other, with the first circle being tangent to $\overline{AB}$ and the last circle being tangent to $\overline{AC}$, as shown. Similarly, 2024 circles of radius 1 can be placed tangent to $\overline{BC}$ in the same manner. The inradius of $\triangle ABC$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.



Solution:

Answer (197):

Let I be the incenter of $\triangle ABC$, and let h be its inradius, which is the height of $\triangle IBC$ from I to $\overline{BC}$. If there are k circles of radius r tangent to $\overline{BC}$ and tangent to each other in a line, then the distance between the centers of the first and last circles is $2(k-1)r$. For the first and last circles to be tangent to $\overline{AB}$ and $\overline{AC}$, respectively, the first circle's center must lie on $\overline{BI}$, and the last circle's center must lie on $\overline{CI}$.



This implies that $\triangle IBC$ contains an inscribed rectangle with dimensions 34 by $14 \cdot 34$ with its longer side along $\overline{BC}$, as well as an inscribed rectangle with dimensions 1 by $4046 \cdot 1$ with its longer side along $\overline{BC}$. There are two similar triangles each with two sides along $\overline{BI}$ and $\overline{CI}$ and a third side that is a side of one of these two inscribed rectangles. Because similar triangles have altitudes with the same ratio as the ratios of their side lengths,

$$\frac{h-1}{h-34} = \frac{4046}{14 \cdot 34} = \frac{17}{2}.$$

Solving this equation gives $h = \frac{192}{5}$. The requested sum is $192 + 5 = 197$.

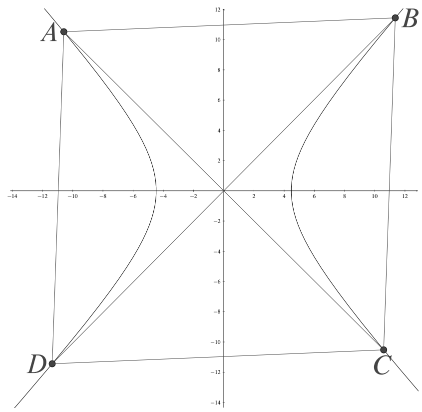
Problem 9:

Let A, B, C, and D be points on the hyperbola $\frac{x^2}{20} - \frac{y^2}{24} = 1$ such that $ABCD$ is a rhombus whose diagonals intersect at the origin. Find the greatest real number that is less than BD^2 for all such rhombi.

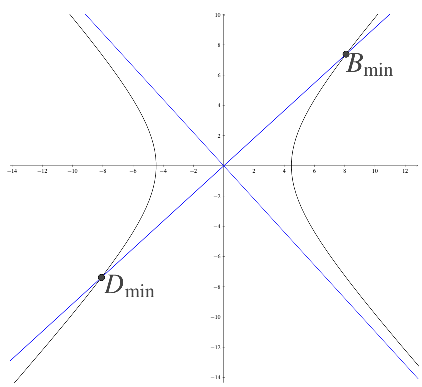
Solution:

Answer (480):

Without loss of generality, let diagonal $\overline{BD}$ have positive slope m, with B having positive coordinates, as in the diagram. Because the diagonals of a rhombus are perpendicular, line AC has slope $-\frac{1}{m}$. If the lines $y = mx$ and $y = -\frac{x}{m}$ each intersect the hyperbola in two points (B, D and A, C, respectively), then $ABCD$ is indeed a rhombus, because symmetry of the hyperbola about the origin implies that $\overline{BD}$ and $\overline{AC}$ bisect each other.



The asymptotes of the hyperbola have slopes $\pm\sqrt{\frac{6}{5}}$, so the necessary intersections occur if both m and $\frac{1}{m}$ are less than $\sqrt{\frac{6}{5}}$, which is equivalent to $\sqrt{\frac{5}{6}} < m < \sqrt{\frac{6}{5}}$. From the diagram, note that as m decreases, the distance from the origin to B decreases, and hence so does BD. Therefore the desired lower bound equals the square of the length of the segment between intersection points of the hyperbola with the line passing through the origin that is perpendicular to the negatively sloped asymptote of the hyperbola, as shown in the diagram below.



The line perpendicular to the asymptote has the equation $y = \sqrt{\frac{5}{6}}x$, which intersects the hyperbola at points $B_{\text{min}}\left(12\sqrt{\frac{5}{11}}, 10\sqrt{\frac{6}{11}}\right)$ and $D_{\text{min}}\left(-12\sqrt{\frac{5}{11}}, -10\sqrt{\frac{6}{11}}\right)$, respectively. The requested square of the distance between these two points is

$$\left(12\sqrt{\frac{5}{11}} + 12\sqrt{\frac{5}{11}}\right)^2 + \left(10\sqrt{\frac{6}{11}} + 10\sqrt{\frac{6}{11}}\right)^2 = \frac{576 \cdot 5 + 400 \cdot 6}{11} = 480.$$

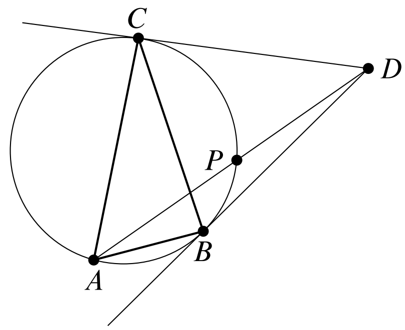
Problem 10:

Let $\triangle ABC$ have side lengths $AB = 5$, $BC = 9$, and $CA = 10$. The tangents to the circumcircle of $\triangle ABC$ at B and C intersect at point D, and $\overline{AD}$ intersects the circumcircle at $P \neq A$. The length of $\overline{AP}$ is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Answer (113):

Note that $\angle DBP = \angle DAB$ and $\angle DCP = \angle DAC$ because the two pairs of angles subtend arcs $\widehat{BP}$ and $\widehat{CP}$, respectively. It follows that $\triangle DBP \sim \triangle DAB$ and $\triangle DCP \sim \triangle DAC$.



Then

$$\frac{AB}{BP} = \frac{AD}{BD} = \frac{AD}{CD} = \frac{AC}{CP}.$$

Because $AB = 5$ and $AC = 10$, it follows that $CP = 2 \cdot BP$. Let $y = BP$; then $CP = 2y$. Applying Ptolemy's Theorem to cyclic quadrilateral $ABPC$ yields

$$\begin{aligned} BC \cdot AP &= AB \cdot CP + AC \cdot BP \\ 9 \cdot AP &= 20y \\ AP &= \frac{20}{9}y. \end{aligned} \tag{1}$$

The problem thus reduces to finding y. Applying the Law of Cosines to $\triangle BPC$ yields

$$\begin{aligned} BP^2 + CP^2 - 2 \cdot BP \cdot CP \cdot \cos(\angle BPC) &= BC^2 \\ 5y^2 - 4y^2 \cos(\angle BPC) &= 81. \end{aligned} \tag{2}$$

Quadrilateral $ABPC$ is cyclic, implying that $\angle BPC = 180° - \angle BAC$, so $\cos(\angle BPC) = -\cos(\angle BAC)$. Applying the Law of Cosines to $\triangle BAC$ yields

$$\cos(\angle BAC) = \frac{5^2 + 10^2 - 9^2}{2 \cdot 5 \cdot 10} = \frac{11}{25}.$$

Substituting this into (2) yields

$$5y^2 + \frac{44}{25}y^2 = 81 \quad \Longrightarrow \quad y = \frac{45}{13}.$$

Substituting this value into (1) yields

$$AP = \frac{20}{9} \cdot \frac{45}{13} = \frac{100}{13}.$$

The requested sum is $100 + 13 = 113$.

OR

Let $\mathcal{T}$ be the transformation of the plane that consists of an inversion of the plane with respect to the circle with center A and radius $\sqrt{AB \cdot AC}$ followed by a reflection of the plane across the angle bisector of $\angle BAC$. Note that $\mathcal{T}$ is an involution, that is, it is its own inverse. Then $\mathcal{T}(B) = C$ and $\mathcal{T}(C) = B$. Line BC goes to a circle ω passing through A, $\mathcal{T}(B)$, and $\mathcal{T}(C)$, so it is the circumcircle of $\triangle ABC$. Thus $\mathcal{T}(BC) = \omega$ and $\mathcal{T}(\omega) = BC$. Line BD is tangent to ω, so $\mathcal{T}(BD)$ is a circle passing through A and C and is tangent to line BC. Similarly, $\mathcal{T}(CD)$ is a circle passing through A and B and is tangent to line BC. Hence $\mathcal{T}(D) = D'$ is the second intersection of the above circles. Line AD' is the radical axis of the two circles, and line BC is tangent to both circles. Therefore AD' intersects $\overline{BC}$ at M, where

$$MC^2 = MD' \cdot MA = MB^2,$$

implying that M is the midpoint of $\overline{BC}$. Thus $\mathcal{T}(P) = \mathcal{T}(\omega \cap AD) = \mathcal{T}(\omega) \cap \mathcal{T}(AD) = BC \cap AD' = M$. This implies that $AP \cdot AM = AB \cdot AC$. Because M is the midpoint of $\overline{BC}$, the distance AM can be found using the Length of Median Formula:

$$AM = \frac{1}{2} \cdot \sqrt{2 \cdot AB^2 + 2 \cdot AC^2 - BC^2} = \frac{1}{2} \cdot \sqrt{2 \cdot 5^2 + 2 \cdot 10^2 - 9^2} = \frac{13}{2}.$$

Then

$$AP = \frac{AB \cdot AC}{AM} = \frac{5 \cdot 10}{\frac{13}{2}} = \frac{100}{13},$$

as above.

OR

As in the first solution, $\frac{AB}{BP} = \frac{AC}{CP}$. Let M be the midpoint of $\overline{BC}$. Applying the Law of Sines to $\triangle AMB$ and $\triangle AMC$ yields

$$BA \cdot \sin(\angle BAM) = BM \cdot \sin(\angle BMA) = CM \cdot \sin(\angle CMA) = CA \cdot \sin(\angle CAM)$$

and, similarly, $BA \cdot \sin(\angle PAC) = CA \cdot \sin(\angle PAB)$. It follows that

$$\frac{\sin(\angle BAM)}{\sin(\angle CAM)} = \frac{\sin(\angle PAC)}{\sin(\angle PAB)}.$$

Note that

$$\frac{\sin(\angle BAM)}{\sin(\angle CAM)} = \frac{\sin(\angle BAC - \angle CAM)}{\sin(\angle CAM)} = \frac{\sin(\angle BAC)\cos(\angle CAM) - \cos(\angle BAC)\sin(\angle CAM)}{\sin(\angle CAM)}$$

$$= \sin(\angle BAC)\cot(\angle CAM) - \cos(\angle BAC).$$

Similarly,

$$\frac{\sin(\angle PAC)}{\sin(\angle PAB)} = \sin(\angle BAC)\cot(\angle PAB) - \cos(\angle BAC).$$

Therefore $\cot(\angle CAM) = \cot(\angle PAB)$ and $\angle CAM = \angle PAB$. Because $\angle ACB = \angle APB$, it follows that $\triangle CAM \sim \triangle PAB$, and $AP \cdot AM = AB \cdot AC$. The remainder follows as in the second solution.

Problem 11:

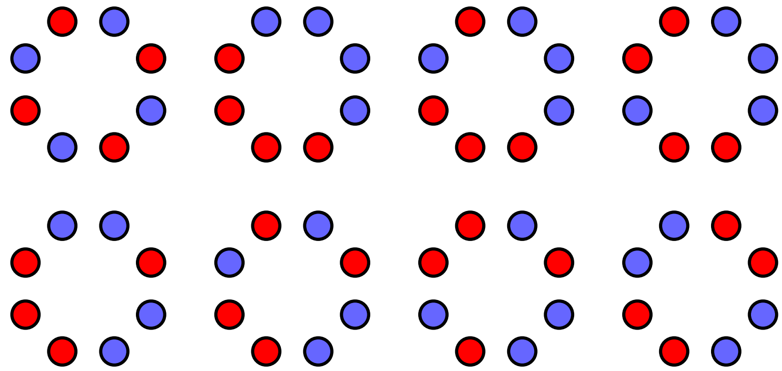
Each vertex of a regular octagon is independently colored either red or blue with equal probability. The probability that the octagon can then be rotated so that all of the blue vertices move to positions where there had been red vertices is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:

Answer (371):

There are $2^8 = 256$ equally likely ways to color the vertices of the octagon. If more than 4 of the vertices are colored blue, then it is not possible to rotate the octagon so that blue vertices move to positions previously occupied by red vertices. If 3 or fewer of the vertices are colored blue, then it is always possible to rotate the octagon so that the blue vertices move to positions where there had been red vertices. Indeed, if there are at most 3 blue vertices, then there are at most 2 rotations that move a particular blue vertex onto a position of a different blue vertex. Thus there are at most $2 + 2 + 2 = 6$ rotations that move at least one blue vertex onto the position of another blue vertex. Because there are 7 possible rotations that move the vertices, there must be at least one rotation that moves all blue vertices to positions where there were red vertices. The number of ways to color 3 or fewer vertices blue is $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} + \binom{8}{3} = 93$.

It is left to count the number of ways to color 4 of the vertices blue and have an acceptable rotation. Each coloring with 4 blue vertices is a rotation or reflection and rotation of one of the following eight patterns: the first with no two blue vertices adjacent, the second with 4 blue vertices adjacent, the third and fourth with 3 blue vertices adjacent, and the last four with 2 blue vertices adjacent.



The first, second, seventh, and eighth of these colorings have acceptable rotations while the other four colorings do not have acceptable rotations. There are 2, 8, 8, and 4 colorings of the vertices similar to the first, second, seventh, and eighth colorings, respectively, giving $2+8+8+4 = 22$ colorings with 4 blue vertices that have acceptable rotations.

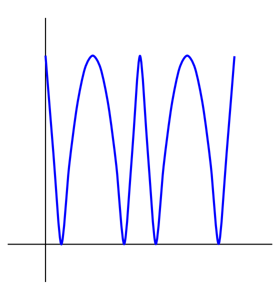
The total number of colorings with acceptable rotations is $93 + 22 = 115$. The required probability is $\frac{115}{256}$. The requested sum is $115 + 256 = 371$.

Problem 12:
Define $f(x) = \left||x| - \frac{1}{2}\right|$ and $g(x) = \left||x| - \frac{1}{4}\right|$. Find the number of intersections of the graphs of

$$y = 4g\big(f(\sin(2\pi x))\big) \qquad \text{and} \qquad x = 4g\big(f(\cos(3\pi y))\big).$$

Solution:

Answer (385):
The graph of $y = \sin(2\pi x)$ on the interval $0 \le x \le 1$ traces one complete period of the sine wave, where y oscillates from 0 to 1 to -1 and back to 0. Thus the graph of $y = |\sin(2\pi x)|$ oscillates 0, 1, 0, 1, 0. It then follows that the graph of $y = 2f(2\sin(2\pi x)) = 2\left||\sin(2\pi x)| - \frac{1}{2}\right|$ oscillates 1, 0, 1, 0, 1, 0, 1, 0, 1, as shown.



Note that this graph passes through $(0, 1)$ and $(1, 1)$, and it zigzags down and up between the lines $y = 1$ and $y = 0$ a total of 4 times in this interval. In particular, there are 8 intersection points between this graph and the line $y = \frac{1}{2}$. Therefore the curve $y = 4g\big(f(\sin(2\pi x))\big)$ goes through $(0, 1)$ and $(1, 1)$, and consists of 8 monotonically decreasing sections which alternate with 8 monotonically increasing sections, where each section has endpoints on the lines $y = 0$ and $y = 1$. Similarly, the curve $x = 4g\big(f(\cos(3\pi y))\big)$ goes through $(1, 0)$ and $(1, 1)$ and consists of 12 monotonically decreasing sections which alternate with 12 monotonically increasing sections, where each section has endpoints on the lines $x = 0$ and $x = 1$. These curves have no intersections outside of the unit square with vertices at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

The only intersection of the two curves on the boundary of the unit square is at the point $(1, 1)$. Because the graph of $y = \sin(2\pi x)$ crosses the line $x = 1$ without being tangent to it, the curve $y = 4g\big(f(\sin(2\pi x))\big)$ meets the line $x = 1$ at $y = 1$ without being tangent to it. However, because the graph of $x = \cos(3\pi y)$ is tangent to the y-axis at $y = 1$, the curve $x = 4g\big(f(\cos(3\pi y))\big)$ is tangent to the line $x = 1$ at $y = 1$. As a result, the rightmost monotonically increasing section of the curve $y = 4g\big(f(\sin(2\pi x))\big)$ intersects the uppermost monotonically increasing section of the curve $x = 4g\big(f(\cos(3\pi y))\big)$ at a point in the interior of the square. Therefore the number of intersections between these curves in the interior of the square is the same as the number of intersections between a set of 16 vertical lines and 24 horizontal lines, which is 384. Because the curves also intersect at $(1, 1)$ on the boundary of the square, the two curves intersect at a total of $384 + 1 = 385$ points.

Problem 13:

Let p be the least prime number for which there exists an integer n such that $n^4 + 1$ is divisible by p^2. Find the least positive integer m such that $m^4 + 1$ is divisible by p^2.

Solution:

Answer (110):

First, note that $p \neq 2$, because $n^4 + 1$ is not divisible by 4 for any integer n. Moreover, if $n^4 + 1 \equiv 0 \pmod{p}$, then $n^4 \equiv -1 \pmod{p}$ and $n^8 \equiv 1 \pmod{p}$, so n has order 8, implying that $8 \mid (p - 1)$. The least prime that satisfies this condition is $p = 17$. So, consider whether there is an integer n such that 17^2 divides $n^4 + 1$.

Let $m = 17k + r$, with $0 \le r \le 16$. Then

$$m^4 + 1 = (17k + r)^4 + 1 = 17^2\left(17^2k^4 + 4 \cdot 17k^3r + 6k^2r^2\right) + 4 \cdot 17kr^3 + r^4 + 1$$

is divisible by 17^2 if and only if $4k \cdot r^3 + \frac{r^4+1}{17}$ is divisible by 17. The solutions to $r^4 + 1 \equiv 0 \pmod{17}$ are $r = 2$, 8, 9, and 15. For each r, the least values of k and m are given by the following ordered triples

$$(r, k, m) = (2, 9, 155), (8, 6, 110), (9, 10, 179), (15, 7, 134).$$

Thus the least possible prime p is 17, and the least positive integer m for which 17^2 divides $m^4 + 1$ is 110.

Problem 14:

Let $ABCD$ be a tetrahedron such that $AB = CD = \sqrt{41}$, $AC = BD = \sqrt{80}$, and $BC = AD = \sqrt{89}$. There exists a point I inside the tetrahedron such that the distances from I to each of the faces of the tetrahedron are all equal. This distance can be written in the form $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime. Find $m + n + p$.

Solution:

Answer (104):

The required distance is the radius, r, of the insphere of the tetrahedron. Note that $\triangle ABC$, $\triangle BAD$, $\triangle CDA$, and $\triangle DCB$ are congruent, and thus they have the same area. Let this shared area be K. This implies that the distances from any one of the vertices to the base containing the other three vertices must all be equal. Let this value be h. The volume of tetrahedron $ABCD$ is then $\frac{1}{3}hK$, and it is also equal to the sum of the volumes of tetrahedra $ABCI$, $ABDI$, $ACDI$, and $BCDI$. These tetrahedra have a base of area K and a height r, so the volume of tetrahedron $ABCD$ is also $4 \cdot \frac{1}{3}rK$. Therefore $\frac{1}{3}hK = 4 \cdot \frac{1}{3}rK$, so $r = \frac{h}{4}$.

Let u, v, and w be positive real numbers that satisfy the equations

$$\begin{aligned} AB^2 &= 41 = u^2 + v^2, \\ AC^2 &= 80 = v^2 + w^2, \text{ and} \\ BC^2 &= 89 = u^2 + w^2. \end{aligned}$$

Placing points A, B, and C in 3-dimensional coordinate space at $A(0, 0, 0)$, $B(u, v, 0)$, $C(0, v, w)$ results in segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ having the required lengths. In addition, if point D is chosen to be $D(u, 0, w)$, then

$$\begin{aligned} AB^2 = CD^2 &= 41 = u^2 + v^2, \\ AC^2 = BD^2 &= 80 = v^2 + w^2, \text{ and} \\ BC^2 = AD^2 &= 89 = u^2 + w^2, \end{aligned}$$

and $ABCD$ is the required tetrahedron. The system above has a unique solution $u = 5$, $v = 4$, and $w = 8$, and thus the vertices of the tetrahedron are $A(0, 0, 0)$, $B(5, 4, 0)$, $C(0, 4, 8)$, and $D(5, 0, 8)$. Vertices B, C, and D lie in the plane with equation

$$\frac{x}{5} + \frac{y}{4} + \frac{z}{8} = 2.$$

The distance from a point (x, y, z) to the plane with equation $ax + by + cz + d = 0$ is given by

$$\frac{|ax + by + cz + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

The altitude of the tetrahedron is the distance from A to the plane containing B, C, and D, which is

$$h = \frac{2}{\sqrt{\frac{1}{5^2} + \frac{1}{4^2} + \frac{1}{8^2}}} = \frac{80\sqrt{21}}{63}.$$

The required inradius of the tetrahedron is $\frac{h}{4} = \frac{20\sqrt{21}}{63}$. The requested sum is $20 + 21 + 63 = 104$.

OR

As in the first solution, three of the edges of the tetrahedron can be represented by the vectors $\langle 5,4,0\rangle$, $\langle 0,4,8\rangle$, and $\langle 5,0,8\rangle$. The area of one of the faces of the tetrahedron can be calculated using Heron's Formula or by calculating one half the length of the cross product of two of these vectors:

$$\frac{1}{2}\cdot\left|\langle 5,4,0\rangle\times\langle 0,4,8\rangle\right| = 6\sqrt{21}.$$

The given tetrahedron fits in a rectangular box with dimensions $5\times 4\times 8$, and the volume of the tetrahedron is the volume of this box minus the volumes of four congruent corners. The volume of each corner is $\frac{1}{3}\cdot\frac{1}{2}\cdot 5\cdot 4\cdot 8$, so the volume of the tetrahedron is

$$5\cdot 4\cdot 8 - 4\cdot\frac{5\cdot 4\cdot 8}{6} = \frac{160}{3}.$$

The volume of the tetrahedron can also be calculated as one sixth the triple product of the vectors:

$$\frac{1}{6}\cdot\begin{vmatrix}5 & 4 & 0\\ 0 & 4 & 8\\ 5 & 0 & 8\end{vmatrix} = \frac{160}{3}.$$

As in the first solution, the required distance, r, is one quarter of the altitude of the tetrahedron, so the volume of the tetrahedron is

$$\frac{1}{3}\cdot 4r\cdot 6\sqrt{21} = \frac{160}{3},$$

from which $r = \frac{20\sqrt{21}}{63}$, as in the first solution.

Note: The volume of a tetrahedron with known edge lengths can also be calculated using Tartaglia's Formula. Also, the square of the volume of an isosceles tetrahedron with side lengths a, b, and c can be written as $\frac{1}{72}\left(a^2+b^2-c^2\right)\left(b^2+c^2-a^2\right)\left(c^2+a^2-b^2\right)$.

Problem 15:

Let $\mathcal{B}$ be the set of rectangular boxes with surface area 54 and volume 23. Let r be the radius of the smallest sphere that can contain each of the rectangular boxes that are elements of $\mathcal{B}$. The value of r^2 can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution:

Answer (721):

Consider a rectangular box with dimensions $a\times b\times c$ from $\mathcal{B}$. From the given conditions $ab+bc+ca=27$ and $abc=23$. The diagonal of the box is $\sqrt{a^2+b^2+c^2}$. The maximum value of $\sqrt{a^2+b^2+c^2}$ among all possible values is the diameter of the smallest sphere that can contain each box from $\mathcal{B}$. This is equivalent to maximizing $s=a+b+c$, because $(a+b+c)^2=\left(a^2+b^2+c^2\right)+2(ab+bc+ca)=a^2+b^2+c^2+54$.

Consider $f(x) = (x-a)(x-b)(x-c) = x^3 - sx^2 + 27x - 23$. The problem is then equivalent to finding the maximum possible s for which this cubic has three positive real roots (allowing multiplicity). Because $uv + vw + wu \le u^2 + v^2 + w^2$ for all real numbers u, v, and w, it follows that $(u+v+w)^2 \ge 3(uv+vw+wu)$ and thus

$$(a+b+c)^2 \ge 3(ab+bc+ca) \qquad \text{and} \qquad (ab+bc+ca)^2 \ge 3abc(a+b+c),$$

which together imply that $9 \le s \le \frac{243}{23}$. When $s = 9$ or $s = \frac{243}{23}$, equality must hold in the above inequalities, implying both $a = b = c = 3$ and $abc = 23$. Thus the cubic does not have three real roots (allowing multiplicity) for s at or near 9 or $\frac{243}{23}$.

As s increases across the interval $\left(9, \frac{243}{23}\right)$, the graph of the cubic curve moves down. There are two limiting cases with three positive real roots. When s is the greatest possible, the local maximum will be a double root (which will be less than the third root). When s is the least possible, the local minimum will be a double root (which will be greater than the third root). If the three roots are a, b, b, then $b^2 + 2ab = 27$ and $ab^2 = 23$. Hence $b^2 + \frac{46}{b} = 27$, and b is a root of $b^3 - 27b + 46 = (b-2)\left(b^2 + 2b - 23\right)$, so $b = 2$ and $b = -1 \pm 2\sqrt{6}$.

The double root $-1 - 2\sqrt{6}$ is not positive, so consider two boundary cases with roots

$$2, 2, \frac{23}{4} \qquad \text{and} \qquad 2\sqrt{6}-1, 2\sqrt{6}-1, \frac{23}{\left(2\sqrt{6}-1\right)^2}.$$

The former triple of roots yields the greatest value for s because the value of the double root is less than the third root. Thus the maximum of $a^2 + b^2 + c^2$ occurs when $a = \frac{23}{4}$ and $b = c = 2$. In that case $r^2 = \frac{1}{4} \cdot \left(a^2 + b^2 + c^2\right) = \frac{657}{64}$. The requested sum is $657 + 64 = 721$.

OR

A box in $\mathcal{B}$ has dimensions (x, y, z), where x, y, z are positive real numbers satisfying

$$xyz = 23 \quad \text{and} \tag{1}$$

$$2xy + 2yz + 2zx = 54. \tag{2}$$

As in the first solution, it is sufficient to find (x, y, z) that maximizes $x+y+z$. Because $2xy+2yz+2zx = 54$, the dimensions cannot all be less than 3, so assume $x \ge 3$. The value of x determines both yz and $y+z$:

$$yz = \frac{23}{x} = P \quad \text{and} \tag{3}$$

$$y + z = \frac{27}{x} - \frac{23}{x^2} = S. \tag{4}$$

This system has positive solutions for y and z if P and S are both positive and satisfy $S^2 - 4P \geq 0$. When $x \geq 3$, both P and S are positive, so x is the dimension of a box in $\mathcal{B}$ if and only if

$$\left(\frac{27}{x} - \frac{23}{x^2}\right)^2 - 4\left(\frac{23}{x}\right) \geq 0,$$

which is equivalent to

$$g(x) = -92x^3 + 27^2x^2 - (2 \cdot 23 \cdot 27)x + 23^2 \geq 0.$$

Note that $g\left(\frac{1}{2}\right) > 0$, $g(1) < 0$, $g(3) > 0$, and $g(10) < 0$. It follows that this cubic has three real roots, all in the interval $\left(\frac{1}{2}, 10\right)$. The Rational Root Theorem suggests that 1 and $\frac{23}{4}$ might be roots in the interval $\left(\frac{1}{2}, 10\right)$, and indeed, $\frac{23}{4}$ is a root. It follows that $g(x) \geq 0$ when $x \in \left[3, \frac{23}{4}\right]$, and these are the values of x that correspond to boxes in $\mathcal{B}$.

To find the value of x in $\left[3, \frac{23}{4}\right]$ that maximizes

$$f(x) = x + y + z = x + \frac{27}{x} - \frac{23}{x^2},$$

note that

$$f\left(\frac{23}{4}\right) - f(x) = \frac{1}{x^2} \cdot \left(x - \frac{23}{4}\right)(x-2)^2,$$

which is 0 at $\frac{23}{4}$ and positive elsewhere in $\left[3, \frac{23}{4}\right]$. In this case $y = z = 2$ and

$$r^2 = \frac{1}{4}\left(x^2 + y^2 + z^2\right) = \frac{657}{64},$$

as in the first solution.

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